

# Cosmic strings and strings in gravitational waves

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## Abstract

We consider strings with the Nambu action as extremal surfaces in a given space-time, thus, we ignore their back reaction. Especially, we look for strings sharing one symmetry with the underlying space-time. If this is a non-null symmetry, the problem of determining the motion of the string can be dimensionally reduced. We get exact solutions for the following cases: straight and circle-like strings in a Friedmann background, straight strings in an anisotropic Kasner background, different types of strings in the metric of a gravitational wave. The solutions will be discussed.

Wir betrachten Strings mit der Nambu-Wirkung als Extremalflächen in einer gegebenen Raum-Zeit, d.h., wir ignorieren ihre Rückwirkung. Wir interessieren uns dabei besonders für solche Strings, die eine Isometrie mit der unterliegenden Raum-Zeit gemeinsam haben. Handelt es sich dabei um eine nicht-lichtartige Symmetrie, so lässt sich das Problem der Bestimmung der Stringbewegung dimensionsreduzieren. Wir erhalten exakte Lösungen für die folgenden Fälle: gerade und

kreisrunde Strings im Friedmann-Hintergrund, gerade Strings im anisotropen Kasner-Hintergrund sowie verschiedene Stringtypen in der Metrik einer Gravitationswelle. Die Lösungen werden diskutiert.

Key words: gravitation theory, strings, gravitational waves

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## 1 Introduction

To give detailed arguments for considering strings means carrying coals to Newcastle. So we only list the main points: A string is, generally speaking, an object possessing a two-dimensional world surface (= world sheet) in contrast to a point particle possessing a one-dimensional world line, cf. the review article VILENKO (1985). In details

1. One considers strings with the Nambu action in a  $D$ -dimensional flat space-time. The theory can be consistently quantized for  $D = 26$  only: otherwise the light cone quantization leads to a breakdown of Lorentz covariance, cf. GREEN, SCHWARZ and WITTEN (1987). But we consider a classical (= non-quantized) theory only and require  $D = 4$  henceforth.
2. Cosmic strings are one-dimensional topological defects in gauge field theories. They have a large mass and can be seeds for larger objects by accretion processes, cf. ZELDOVICH (1980).
3. One looks for solutions of the Einstein equation with distribution-valued energy-momentum tensor whose support is a two-dimensional submanifold of indefinite signature. The equation of state is  $p_z = -\mu$  and the solution is called cosmic string. These are good candidates for seeds of galaxies in the early universe.
4. If one looks for strings according to 2. or 3. in a given space-time, i.e., with negligible back reaction of the string onto space-time geometry, one arrives at the Nambu action, too, see NIELSEN and OLESEN (1973) for

2. and GEROCH and TRASCHEN (1987) for 3. In other words, cosmic strings are extremal (i.e., maximal or minimal in dependence of the boundary conditions) surfaces of indefinite signature in a given space-time. This approach is justified if the diameter of the string is small compared with the curvature radius of the underlying manifold and if for the string tension  $\alpha' \ll 1/G$  holds (we use  $c = 1$ ). Normally, one thinks in orders of magnitude  $G\alpha' = 10^{-6 \pm 2}$ , see BRANDENBERGER (1987). In other words, the string is supposed to possess a mass per unit length of  $10^{22 \pm 2}$  g/cm, if the phase transition is supposed to be at the GUT-scale (see FROLOV and SEREBRIANY 1987).

In FROLOV et al. (1988) a stationary string in the Kerr-Newman metric has been considered. We use the method developed there and apply it to other cases.

In the present paper we shall adopt the 4. approach and try to give some geometrical insights into the motion of a string. To this end we give a sample of closed-form solutions for a special class of string solutions: strings which share a space-like or timelike isometry with the underlying background metric.

The paper is organized as follows: sct. 2 contains the main formulae, sct. 3 the string in a Friedmann background, sct. 4 the string in the anisotropic Kasner background and sct. 5 the string in a gravitational wave.

## 2 The main formulae

The string is a two-dimensional extremal surface of indefinite signature. Let us take coordinates  $(\tau, \sigma) = (y^A)$ ,  $A = 0, 1$  within the string and coordinates  $(x^i)$ ,  $i = 0, 1, 2, 3$  for the space-time  $V_4$  with the metric  $g_{ij}$ . The string is

given by specifying the four functions  $x^i(y^A)$ . The signature for the metric  $g_{ij}$  is  $(+ - - -)$ . The tangents to the string are

$$e_A^i = \partial x^i / \partial y^A \quad (1)$$

and induce the metric

$$h_{AB} = e_A^i e_B^j g_{ij} \quad (2)$$

at the string world sheet. The signature of the string is required to be  $(+-)$ , thus

$$h \equiv \det h_{AB} < 0. \quad (3)$$

The action to be varied is

$$I = \frac{1}{2\alpha'} \int \int \sqrt{-h} d\sigma d\tau. \quad (4)$$

Instead of writing down the full equations we specialize to the case we are interested here: we require that an isometry for both the underlying space-time and the string exists. Let  $k_i$  be a non-null hypersurface-orthogonal Killing vector field, i.e.,

$$k_i k^i \neq 0, \quad k_{[i} k_{j;k]} = 0, \quad k_{(i;j)} = 0. \quad (5)$$

Then there exist coordinates  $x^i$  ( $x^0 = t$ ) such

$$ds^2 = g_{00} dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta. \quad (6)$$

$\alpha, \beta = 1, 2, 3$ , the  $g_{ij}$  do not depend on  $t$ ,  $k_i = \partial/\partial t$ . The sign of  $g_{00}$  is determined by the condition  $g_{00} k_i k^i > 0$ : for timelike  $k_i$  we have  $g_{00} > 0$  and for spacelike  $k_i$  we have  $g_{00} < 0$ . The requirement that  $k_i$  is also an isometry of the string gives us the possibility to specify the functions  $x^i(y^A)$  to be  $t = \tau$ ,  $x^\alpha$  depends on  $\sigma$  only. Then we get with (1)

$$e_0^i = (1, 0, 0, 0), \quad e_1^i = (0, dx^\alpha / d\sigma). \quad$$

With (2, 4) we get  $h_{01} = 0$ ,  $h_{00} = g_{00}$ ,

$$h_{11} = g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma},$$

$$I = \frac{1}{2\alpha'} \int \int \sqrt{-g_{00} g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}} d\sigma d\tau. \quad (7)$$

The integrand does not depend on  $\tau$ , so we can omit the  $\tau$  integration. Therefore, extremizing (7) is the same as solving the geodesic equation for the auxiliary metric  $f_{\alpha\beta}$  of a  $V_3$  defined by

$$f_{\alpha\beta} = -g_{00}g_{\alpha\beta}. \quad (8)$$

Remarks: 1. If  $k_i$  is not hypersurface-orthogonal then eq. (6) contains terms with  $g_{0\alpha}$  and one has to add  $g_{0\alpha}g_{0\beta}$  to the r.h.s. of eq. (8). 2. The dimensional reduction is not fully trivial: the geodesic equation for (8) corresponds to compare the strings in (7) with other strings sharing the same isometry induced by  $k_i$ , whereas the Nambu action has to be compared with all other strings, too. But writing down all full equations or counting the degrees of freedom one can see that at our circumstances no difference appears.

For a time-like  $k_i$ ,  $f_{\alpha\beta}$  is positive definite, and the condition (3) is automatically fulfilled. On the other hand, for a space-like  $k_i$ ,  $f_{\alpha\beta}$  is of signature  $(+--)$  and eq. (3) requires the vector  $dx^\alpha/d\sigma$  to be time-like, i.e., the root in eq. (7) has to be real.

### 3 The string in a Friedmann model

Now we specify the underlying  $V_4$  to be a spatially flat Friedmann model

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \quad (9)$$

#### 3.1 The open string

First we use the space-like Killing vector  $\partial/\partial x$  of (9). We have in mind an infinitely long straight string moving through the expanding universe. We

apply the formalism of sct. 2 and get the following result: we write eq. (9) in the form of eq. (6)

$$ds^2 = -a^2 dx^2 + dt^2 - a^2(dy^2 + dz^2).$$

The metric (8) then reads

$$ds_{(3)}^2 = a^2 dt^2 - a^4(dy^2 + dz^2).$$

Without loss of generality the string is situated at  $z = 0$  and moves into the  $y$ -direction according to

$$\begin{aligned} ds_{(3)}^2 &= a^2 dt^2 - a^4 dy^2 \\ \ddot{t} + \frac{1}{a} \frac{da}{dt} \dot{t}^2 + 2 \frac{da}{dt} a \dot{y}^2 &= 0 \quad \text{with} \quad \cdot = d/d\lambda \\ a^4 \dot{y} &= M \\ a^2 \dot{t}^2 - a^4 \dot{y}^2 &= 1 \\ y(t) &= M \int \frac{dt}{a(t) \sqrt{a^4(t) + M^2}} \end{aligned} \tag{10}$$

where  $M$  is an integration constant. The natural distance of the string from the origin is

$$s(t) = a(t)y(t).$$

*1. Example.* Let  $a(t) = t^n$ ,  $n \geq 0$  then for  $t \gg 1$ , cf. STEIN-SCHABES and BURD (1988),

$$y(t) \approx Mt^{1-3n}, \quad s(t) \approx Mt^{1-2n} \quad \text{for} \quad n \neq 1/3$$

and  $y(t) \approx M \ln t$  for  $n = 1/3$ .

Interpretation:  $n = 0$ , i.e., the absence of gravity, implies a linear motion as it must be the case. We have

$$\lim_{t \rightarrow \infty} y(t) = \infty$$

if and only if  $n \leq 1/3$ , i.e., for  $n > 1/3$  the string comes to rest with respect to the cosmic background after a finite time. A more stringent condition to be discussed is that the string comes to rest in a natural frame of a suitably chosen reference galaxy. This means

$$\lim_{t \rightarrow \infty} s(t) < \infty$$

and is fulfilled for  $n \geq 1/2$ . Therefore, the most interesting cases  $n = 1/2$  (radiation model) and  $n = 2/3$  (Einstein-de Sitter dust model) have the property that straight open strings come to rest after a sufficiently long time independently of the initial conditions.

2. *Example.* Let  $a(t) = e^{Ht}$ , the inflationary model,  $H > 0$ . Then  $y(t) = e^{-3Ht}$ ,  $s(t) = e^{-2Ht}$ . We have the same result as in the first example with  $n \gg 1$ .

3. *Example.* Let

$$a(t) = -t^{2/3}(1 + t^{-2} \cos mt).$$

This background metric is from damped oscillations of a massive scalar field or, equivalently, from fourth order gravity  $L = R + m^{-2} R^2$ . We get with eq. (10)

$$y(t) \approx -1/t - ct^{-4} \sin mt, \quad c = \text{const.} \quad \text{for } t \rightarrow \infty,$$

which is only a minor modification of the result of the first example with  $n = 2/3$ , therefore, one should not expect a kind of resonance effect between the open string and a massive scalar field.

## 3.2 The closed string

We insert  $dy^2 + dz^2 = dr^2 + r^2 d\Phi^2$  into eq. (9) and use the space-like Killing vector  $\partial/\partial\Phi$ : its trajectories are circles, so we have in mind a closed string

with radius  $r$  moving (and eventually oscillating) in the expanding universe. The corresponding geodesic equation leads to

$$\frac{d}{d\lambda}(-a^4r^2\dot{r}) - a^2r\dot{t}^2 + a^4r(\dot{x}^2 + \dot{r}^2) = 0, \quad (11)$$

$$\frac{d}{d\lambda}(a^4r^2\dot{x}) = 0, \quad (12)$$

$$\dot{t}^2 - a^2(\dot{x}^2 + \dot{r}^2) = \frac{1}{a^2r^2}, \quad (13)$$

where the dot denotes  $d/d\lambda$ ,  $\lambda$  is the natural parameter along the geodesic. We are mainly interested in solutions not moving into the  $x$ -direction, to understand the oscillating behaviour. Inserting  $x = 0$  into eqs. (11 - 13) and using the fact that always  $\dot{t} \neq 0$  holds, eqs. (11-13) can be brought into the form

$$r \cdot r''a^3 - 2rr'a^4a' - r'^2a^3 + 3rr'a^2a' + a = 0, \quad (14)$$

where the dash denotes  $d/dt$ .

*4. Example.* Let  $a = 1$ , i.e., we have the flat Minkowski spacetime. Eq. (14) then reduces to

$$rr'' = r'^2 - 1. \quad (15)$$

The solution reads

$$r(t) = r_0 \cos((t - t_0)/r_0), \quad r' = -\sin((t - t_0)/r_0). \quad (16)$$

This is the oscillating solution for closed strings. At points  $t$  where  $r = 0$ , we have  $|r'| = 1$ , see eq. (16). These points are the often discussed cusp points of the string, where the interior string metric becomes singular and the motion approximates the velocity of light (cf. THOMPSON 1988).

Let us compare eq. (15) with the analogous equation for a positive definite back-ground metric. It reads

$$rr'' = r'^2 + 1$$

and has solutions with  $\cosh$  instead of  $\cos$ . This is then the usual minimal surface taken up e.g. by a soap-bubble spanned between two circles.

5. *Example.* Let  $a(t) = t^n$ , then eq. (14) specializes to

$$rr''t^{2n} - 2nrr'^3t^{4n-1} - r'^2t^{2n} + 3nrr't^{2n-1} + 1 = 0. \quad (17)$$

This equation governs the radial motion of a circle-like closed string in a Friedmann background. The solutions can be hardly obtained by analytic methods.

## 4 The string in an anisotropic Kasner background

Now we take as background metric

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2 \quad (18)$$

and  $\partial/\partial x$  as Killing vector. Astonishingly, the anisotropy has only a minor influence on the motion of the string, so we get mainly the same formulae as in sect. 3.1.: the geodesic equations are

$$\begin{aligned} \frac{d}{d\lambda}(a^2b^2\dot{y}) &= 0 \quad \text{hence} \quad a^2b^2\dot{y} = M_1 \\ \frac{d}{d\lambda}(a^2c^2\dot{z}) &= 0 \quad \text{hence} \quad a^2c^2\dot{z} = M_2 \\ a^2\dot{t}^2 - a^2c^2\dot{z}^2 &= 1 \end{aligned}$$

and can be integrated to yield for  $M_1M_2 \neq 0$

$$y(r) = \int \frac{dt}{b(t)\sqrt{\frac{a^2b^2}{M_1^2} + \frac{M_2^2}{M_1^2} \cdot \frac{b^2}{c^2} + 1}}. \quad (19)$$

The equation for  $z(t)$  can be obtained from eq. (19) by interchanging  $b \leftrightarrow c$  and  $M_1 \leftrightarrow M_2$ .

For the Kasner metric we have eq. (18) with  $a = t^p$ ,  $b = t^q$ ,  $c = t^r$ ,  $p + q + r = p^2 + q^2 + r^2 = 1$ . We get

$$y(r) = \int \frac{dt}{t^q \sqrt{t^{2(p+q)} + t^{2(q-r)} + 1}}.$$

The behaviour for  $t \rightarrow \infty$  in dependence of the values  $p$ ,  $q$ ,  $r$  can be seen from this equation.

## 5 The string in a gravitational wave

As background metric we use the plane-wave ansatz

$$ds^2 = 2dudv + p^2(u)dy^2 + q^2(u)dz^2. \quad (20)$$

Eq. (20) is a solution of Einstein's vacuum equation if

$$qd^2p/du^2 + pd^2q/du^2 = 0. \quad (21)$$

Let us take  $\partial/\partial y$  as Killing vector. Then the geodesic equations for the auxiliary metric are

$$\begin{aligned} \frac{d}{d\lambda}(p^2\dot{u}) &= 0 \quad \text{hence} \quad p^2\dot{u} = M_1 \neq 0, \\ \frac{d}{d\lambda}(p^2q^2\dot{z}) &= 0 \quad \text{hence} \quad p^2q^2\dot{z} = M_2, \\ -2p^2\dot{u}\dot{v} - p^2q^2\dot{z}^2 &= 1. \end{aligned}$$

If we take  $u$  as new independent variable (which is possible because of  $\dot{u} \neq 0$ ) we get the solutions

$$\begin{aligned} z(u) &= \frac{M_2}{M_1} \int \frac{du'}{q^2(u')} , \\ v(u) &= \frac{-M_2^2}{2M_1^2} \int \left( \frac{1}{q^2(u')} - p^2(u') \right) du'. \end{aligned} \quad (22)$$

As an example let us take

$$p(u) = \sin u, \quad q(u) = \sinh u \quad (23)$$

which turns out to be a solution of eq. (21). Inserting (23) into eq. (22) we get

$$z(u) = M \coth u, \quad v(u) = -M^2(4 \coth u + \sin 2u - 2u)/8.$$

Another Killing vector of eq. (20) is  $\partial/\partial v$ , but it is a null Killing vector and so the reduction used above does not work. But there exists a further non-null Killing vector of eq. (20). It reads

$$k_i = (0, y, H, 0), \quad \text{where} \quad H = - \int p^{-2} du.$$

By a coordinate transformation

$$\begin{aligned} u &= t & y &= w \cdot e^{-G} & G(u) &= \int \frac{du}{p^2 H} \\ z &= k & v &= x + \frac{1}{2H} e^{-2G} (w^2 - 1) \end{aligned}$$

we get the form (6)

$$ds^2 = e^{-2G} p^2 dw^2 + 2dxdt + \frac{e^{-2G}}{p^2 H^2} dt^2 + q^2 dk^2$$

and have to solve the equations

$$\begin{aligned} \frac{d}{d\lambda} (p^2 q^2 e^{-2G} \dot{k}) &= 0, & \frac{d}{d\lambda} (p^2 e^{-2G} \dot{t}) &= 0, \\ -p^2 q^2 e^{-2G} \dot{k}^2 - \frac{1}{H^2} e^{-4G} \dot{t}^2 - 2p^2 e^{-2G} \dot{x} \dot{t} &= 1. \end{aligned}$$

For the special case  $\dot{k} = 0$  one has finally

$$x(t) = -\frac{1}{2} \int e^{-2G} \left( \frac{p^2}{M_2^2} + \frac{1}{H^2 p^2} \right) dt.$$

## 6 Discussion

Let us suppose a spatially flat Friedmann model with scale factor  $a(t) = t^n$ , and  $n = 2/3$  (Einstein-de Sitter dust model) or  $n = 1/2$  (radiation model).

There we consider an open string which is only a little bit curved such that the approximation of a straight string is applicable. At time  $t = t_0 > 0$  we can prescribe place and initial velocity  $v_0$  of the string arbitrarily and get for  $t \rightarrow \infty$  the behaviour  $s(t) \approx \text{const} + M(v_0)t^{1-2n}$  where  $s$  denotes the natural distance from the origin. That means, for the cases  $n = 2/3$  and  $= 1/2$  we are interested in, the string comes to rest for large values  $t$  at a finite distance from the origin.

We compare this result with the analogous motion of a point-particle in the same background: in the same approximation we get  $s(t) \approx \text{const} + M(v_0)t^{1-n}$ , i.e.,  $|s| \rightarrow \infty$  as  $t \rightarrow \infty$ , a totally other type of motion. It should be mentioned that in the absence of gravity, i.e.,  $n = 0$ , both motions are of the same type, but otherwise not.

The remaining calculations above indicate that the interaction of the motion of the string with scalar field oscillations (3. example), with anisotropy (sct. 4), and with gravitational waves (sct. 5) is quite weak, we did not find any type of resonance effects.

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